Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Let a, b, c be sidelengths of a triangle $ABC.$ Let AA_1, BB_1, CC_1 are heights of the triangle and

let $a_p = B_1C_1, b_p = C_1A_1, c_p = A_1B_1$ be sides of pedal triangle. Prove that:

a) $a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 3abc.$

(or $a_p(b^2 + c^2) + b_p(c^2 + a^2) + c_p(a^2 + b^2) = 3abc$)

b) $a_p + b_p + c_p \le s,$ where s is semiperimeter of $ABC.$

Solution.

Firstly we will prove identity

(**1**) $a^3\cos(B - C) + b^3\cos(C - A) + c^3\cos(A - B) = 3abc.$

Let AA_1, BB_1, CC_1 are heights of the triangle. Then,

$$\sum_{cyc} a^3\cos(B - C) = \sum_{cyc} a^2 2R\sin A \cdot \frac{\sin 2B + \sin 2C}{2\sin A} = R\sum_{cyc} a^2(\sin 2B + \sin 2C) =$$

$$2R\sum_{cyc}(a\sin B \cdot a\cos B + a\sin C \cdot a\cos C) = 2R\sum_{cyc}(CC_1 \cdot BC_1 + BB_1 \cdot CB_1) =$$

$$2R\left(\sum_{cyc} CC_1 \cdot BC_1 + \sum_{cyc} BB_1 \cdot CB_1\right) = 2R\left(\sum_{cyc} CC_1 \cdot BC_1 + \sum_{cyc} CC_1 \cdot AC_1\right) =$$

$$2R\sum_{cyc} CC_1 \cdot (BC_1 + AC_1) = 2R\sum_{cyc} CC_1 \cdot AB = 2R\sum_{cyc} 2[ABC] = 12R \cdot [ABC] = 3abc.$$

Let AA_1, BB_1, CC_1 are heights of the triangle and let $a_p = B_1C_1, b_p = C_1A_1, c_p = A_1B_1$ be sides of pedal triangle. Then

$a_p = a\cos A = 2R\sin A\cos A = R\sin 2A, b = b\cos B = R\sin 2B,$

$c_p = c\cos C = R\sin 2C.$

Also we have:

$\sin 2B + \sin 2C = 2\sin(B + C)\cos(B - C) = 2\sin A\cos(B - C) \Leftrightarrow$

(**2**) $\boxed{\cos(B - C) = \dfrac{\sin 2B + \sin 2C}{2\sin A}} \Leftrightarrow$

$$\cos(B - C) = \frac{R\sin 2B + R\sin 2C}{2R\sin A} = \frac{b_p + c_p}{a} \Rightarrow \frac{b_p + c_p}{a} \le 1 \Leftrightarrow$$

(**3**) $\boxed{b_p + c_p \le a}.$

Thus we obtain identity

(**4**) $a_p + b_p + c_p = a\cos(B - C) + b\cos(C - A) + c\cos(A - B)$

and inequality $a_p + b_p + c_p \le s.$